Name: ______________________________

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# Math 113 (Calculus 2) Section 12
# Exam 4

18–20 November 2010

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Instructions:

1. Work on scratch paper will not be graded.
2. For question 1 and questions 10 through 15, show all your work in the space provided. Full credit will be given only if the necessary work is shown justifying your answer. Please write neatly.
3. Questions 2 through 9 are short answer. Fill in the blank with the appropriate answer. You do not need to show your work.
4. Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
5. Simplify your answers. Expressions such as $\ln(1)$, $e^0$, $\sin(\pi/2)$, etc. must be simplified for full credit.
6. Calculators are not allowed.

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For Instructor use only.

| # | Possible | Earned | | # | Possible | Earned |
|---|---|---|---|---|---|---|
| 1a | 5 | | | 6 | 5 | |
| 1b | 5 | | | 7 | 5 | |
| 1c | 5 | | | 8 | 5 | |
| 1d | 5 | | | 9 | 5 | |
| 1e | 5 | | | 10 | 5 | |
| 1f | 5 | | | 11 | 5 | |
| 2 | 5 | | | 12 | 5 | |
| 3 | 5 | | | 13 | 5 | |
| 4 | 5 | | | 14 | 5 | |
| 5 | 5 | | | 15 | 5 | |
| | | | | Total | 100 | |

Unless indicated, each problem is worth 5%.

1. (30% Show your work.) Determine whether each series converges absolutely, converges conditionally, or fails to converge. State and justify your conclusion next to the series.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$

   Does not converge absolutely by Comparison Test with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$

   Converges conditionally by the Alternating Series Test.

   (b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

   Converges absolutely by the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k^3}}{k^2+1}$

   Does not converge absolutely by Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Converges conditionally by the Alternating Series Test.

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

   Converges absolutely by the root test. The limit of the $n^{th}$ root goes to zero.

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

   Converges absolutely by the Ratio Test. The limit of the ratios goes to zero.

   (f) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n n!}{n^n}$

   Converges absolutely by the Ratio Test. The limit ratio is $\dfrac{1}{e}$.

Questions 2 through 9 are short answer. Fill in the blank with the appropriate answer. You do not need to show your work.

2. If $f(x) = x^3 \cos x$, find the 100th derivative evaluated at zero; i.e., find $f^{(100)}(0)$.

$f(x) = x^3 \cos x = x^3 - \dfrac{x^5}{2!} + \dfrac{x^7}{4!} - \cdots + \dfrac{x^{99}}{96!} - \dfrac{x^{101}}{98!} + \cdots$. Since the coefficient of $x^{100}$ is zero, $f^{(100)}(0) = 0$

3. Evaluate the sum. $1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \dfrac{\pi^6}{6!} + \dfrac{\pi^8}{8!} - \cdots = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!}$ $\qquad \cos \pi = -1$

4. Find the radius of convergence. $\displaystyle\sum_{n=1}^{\infty} \frac{nx^{2n}}{25^n}$ $\qquad r = 5$

5. Find the coefficient of $x^4$ in the Maclaurin series for $\dfrac{\cos x}{1 + x^2}$. $\qquad \dfrac{37}{24}$

6. Find the the first four non-zero terms of Taylor series for $\sin x$ about $x = \pi/6$.

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2 \cdot 2}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2 \cdot 3!}\left(x - \frac{\pi}{6}\right)^3$$

7. Find the Taylor series for $p(x) = x^3 + x^2 + x + 1$ about $x = -1$; i.e., write $p(x)$ as a polynomial in $x + 1$.

$$p(x) = 2(x + 1) - 2(x + 1)^2 + (x + 1)^3$$

8. Find the Maclaurin series for $\dfrac{3}{2+x}$ and give the interval of convergence.

$$\frac{3}{2}(1 - \frac{x}{2} + \frac{x^2}{2^2} - \frac{x^3}{2^3} + \cdots) \quad \text{on} \quad (-2, 2)$$

9. The Taylor polynomial of degree 4 for $f(x) = e^x$ expanded about $a = 0$ is $T_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$. Use the Taylor Inequality to estimate the accuracy of the approximation $f(x) \approx T_4(x)$ when x lies in the interval [-0.5, 0.5].

$$|\text{error}| \leq \frac{\sqrt{e}}{2^5 \cdot 5!}$$

Problems 10-15. Show your work for full credit.

10. Find the sum $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2 + 4k + 3}$

$$\frac{2}{k^2+4k+3} = \frac{1}{k+1} - \frac{1}{k+3}$$

$$S_n = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+1} - \frac{1}{n+3}$$

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \text{ and } \lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

11. Evaluate the sum. $2 \cdot 1x^2 + 3 \cdot 2x^3 + 4 \cdot 3x^4 + 5 \cdot 4x^5 + \cdots = \displaystyle\sum_{n=2}^{\infty} n(n-1)x^n$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3} = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots$$

$$\frac{2x^2}{(1-x)^3} = 2 \cdot 1x^2 + 3 \cdot 2x^3 + 4 \cdot 3x^4 + 5 \cdot 4x^5 + \cdots$$

12. Evaluate the following limit: $\lim\limits_{x \to 0} \dfrac{x^6}{e^{x^3} - 1 - x^3}$

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ so $e^{x^3} = 1 + x^3 + \dfrac{x^6}{2!} + \dfrac{x^9}{3!} + \cdots$

$$\lim_{x \to 0} \frac{x^6}{e^{x^3} - 1 - x^3} = \lim_{x \to 0} \frac{x^6}{1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots - 1 - x^3} = \lim_{x \to 0} \frac{x^6}{\frac{x^6}{2!} + \frac{x^9}{3!} + \cdots} =$$

$$\lim_{x \to 0} \frac{1}{\frac{1}{2!} + \frac{x^3}{3!} + \cdots} = \frac{1}{\frac{1}{2}} = 2$$

13. Find the the first four non-zero terms of a power series expansion for the function $\sin^{-1} x$ expanded about $x = 0$.

Let $y = \sin^{-1} x$. Then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-\frac{1}{2}}$ By the Binomial Theorem

$(1 - x^2)^{-\frac{1}{2}} = 1 + \dfrac{\left(-\frac{1}{2}\right)}{1!}(-x^2) + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots$ So

$\dfrac{dy}{dx} = 1 + \dfrac{1}{2}x^2 + \dfrac{1 \cdot 3}{2^2 2!}x^4 + \dfrac{1 \cdot 3 \cdot 5}{2^3 3!}x^6 \cdots$ and

$y = C + 1x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2^2 2!}\dfrac{x^5}{5} + \dfrac{1 \cdot 3 \cdot 5}{2^3 3!}\dfrac{x^7}{7} \cdots$ But $C = 0$ because $\sin^{-1} 0 = 0$.

14. Find the interval of convergence. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x - 1)^n}{n^2 2^n}$

By the Ratio Test, the series converges for $\dfrac{|x - 2|}{2} < 1$ or $-\dfrac{1}{2} < x < \dfrac{3}{2}$. The series converges absolutely at the endpoints because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. So the interval of convergence is $\left[-\dfrac{1}{2}, \dfrac{3}{2}\right]$.

15. Evaluate the indefinite integral $\displaystyle\int \frac{\sin x}{x} dx$ as an infinite series.

$$\int \frac{\sin x}{x} dx = \int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) dx = C + x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$